(Pages : 3) **T – 6378**

Reg. No. :

Name : [redacted]

Second Semester M.Sc. Degree Examination, September 2024

Statistics

ST 521 : PROBABILITY THEORY — II

(2021 Admission Onwards)

Time : 3 Hours Max. Marks : 60

PART – A

Answer any **four** questions. Each question carries **3** marks.

1. Prove that $\mu'_{(r)} = P_X^r\ (s)$ at $s = 1$, where $\mu'_{(r)}$ is the r[th] descending moment of a r.v X and $P_X^r\ (s)$ is the r[th] derivative of the probability generating function of X.

2. If $g\ (X)$, is a non-negative Borel function of the r.v X, then prove that $P\left(g(X) \geq \epsilon\right) \leq \frac{E\ (g(X))}{\epsilon}$, if $E\left(g(X)\right) < \infty$.

3. State and prove the Jensen's inequality, mentioning the necessary conditions.

4. Prove or disprove: $X_n \xrightarrow{a.s} X \longleftrightarrow X_n \xrightarrow{P} X$.

5. State Helly- Bray theorem.

6. Show that, $1 + \phi_X\ (2t) \geq 2\ (\phi_X\ (t))^2$, where $\phi_X\ (t)$ is the characteristic of X.

P.T.O.

7. Examine whether WLLN holds for the sequence $\{X_n, n \geq\}$ of i.i.d random variables defined as $P\left(X_n = \pm 2^n\right) = 2^{-(2n+1)}$; $n \geq 1$ and $P(X_n = 0) = 1 - 2^{-2n}$.

8. State the Linberg- Feller form of central limit theorem.

(4 × 3 = 12 Marks)

PART – B

Answer any **three** questions. Each question carries **8** marks.

9. For an integer valued r.v X, with $P(X=n) = p_n$ and $P(X \leq n) = q_n$ so that $\sum_{i=0}^{n} p_i = q_n$, then prove that $\sum_{n=0}^{\infty} P(X \leq n)\, s^n = \frac{P_X(s)}{1-s}$, $|s| \leq 1$, where $P_X(s)$ is the probability generating function of X.

10. State and prove the C_r inequality for both $r < 1$ and $r \geq 1$.

11. If X is a non-negative r.v with distribution function $F_X(x)$, then prove that

$$E(X) < \infty \leftrightarrow \int_0^{\infty} (1 - F_X(x)\, dx < \infty \text{ with } E(X) = \int_0^{\infty} (1 - F_X(x)\, dx.$$

12. Prove or disprove: $X_n \xrightarrow{P} X \longrightarrow X_n \xrightarrow{L} X$. Is the converse true? Justify.

13. Define the term convergence in r[th] mean of a sequence of r.vs. Show that $X_n \xrightarrow{r} X \Rightarrow E|X_n|^r \longrightarrow E|X|^r$.

14. Prove the theorem: If $X_n \xrightarrow{L} X$; $Y_n \xrightarrow{L} K$, a constant, then

(a) $X_n + Y_n \xrightarrow{L} X + K$,

(b) $X_n\, Y_n \xrightarrow{L} KX$ and

(c) $\frac{X_n}{Y_n} \xrightarrow{L} \frac{X}{K}$, $K \neq 0$.

(3 × 8 = 24 Marks)

PART – C

Answer any **three** questions. Each question carries **8** marks.

15. Show that characteristic function of a random variable is non-negative definite.

16. (a) State the Fourier inversion theorem of characteristic functions.

    (b) Use it to find the distribution of the random variable whose characteristic function is $\phi_X(t) = e^{-|t|}$, t is real.

17. (a) Prove that the characteristic function is real and even if and only if the corresponding probability distribution is symmetric about the origin.

    (b) State Brochner's Theorem on Fourier Transforms.

18. (a) State and prove the Chebychef's WLLN.

    (b) State Kolmogrov's Three series theorem.

19. (a) State and prove the Lindberg-Levy CLT.

    (b) Show that it is a special case of the Lindberge-Feller form of CLT.

20. (a) Show that every sequence of independent random variables with uniformly bounded variance obeys the SLLN.

    (b) Let the sequence $\{X_n\}$ of independent r.vs with $P[X_n = \pm 1] = \frac{1-2^{-n}}{2}$ and $P[X_n = \pm 2^n] = 2^{-n-1}$. Check whether WLLN and SLLN hold for the sequence.

(3 × 8 = 24 Marks)

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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, November 2023

Statistics

ST 521 : PROBABILITY THEORY – II

(2021 Admission Onwards)

Time : 3 Hours Max. Marks : 60

SECTION – A

Answer any **four** questions. Each question carries **3** marks.

1. Examine whether $P(s) = \frac{2}{1+s}; |s| < 1$, is a probability generating function.

2. State the Markov inequality and deduce the Chebyshev's inequality from it.

3. State Helly-Bray lemma.

4. If $f(x)$ is a continuous function and $X_n \xrightarrow{P} X$, then show that $f(X_n) \xrightarrow{P} f(X)$.

5. Show that the characteristic function exists for all the random variables.

6. Examine whether $\phi(t) = \log(1+t)$ is a characteristic function.

P.T.O.

7. Prove that the WLLN holds for the sequence of random variables $\{X_n, n \geq 1\}$ defined as $P\left(X_n = \pm n^{\alpha}\right) = \frac{1}{2}$ if and only if $\alpha < \frac{1}{2}$.

8. State the Liaponov's form of central limit theorem.

(4 × 3 = 12 Marks)

SECTION – B

Answer any **three** questions. Each question carries **8** marks.

9. If X is a non-negative r.v. with distribution function $F_x(x)$, then $E(X) < \infty \leftrightarrow \int_0^{\infty} [1 - F_x(x)]\, dx < \infty$ and then $E(X) = \int_0^{\infty} [1 - F_x(x)]\, dx$.

10. State and prove the *Cr*- inequality.

11. Prove that $X_n \xrightarrow{P} 0 \leftrightarrow E\left[\frac{|X_n|}{1 + |X_n|}\right] \rightarrow 0$.

12. If $\{X_n, n \geq 1\}$ is a sequence of random variables defined on a probability space (Ω, $\mathfrak{B}$, P), then prove that if $X_n \xrightarrow{a.s} X$, then $X_n \xrightarrow{P} X$. Is the converse true? Justify.

13. State and prove the Slutsky's theorem.

14. Prove that $X_n \xrightarrow{r} X \Rightarrow X_n \xrightarrow{P} X$. Conversely, $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{r} X$ if X_n's are bounded a.s.

(3 × 8 = 24 Marks)

SECTION – C

Answer any **three** questions. Each question carries **8** marks.

15. Establish the elementary properties of a characteristic function.

16. How do you get the moments of a distribution from its characteristic function? Give any one example.

17. State the Fourier inversion theorem of characteristic functions. Use it to find the distribution of the random variable whose characteristic function is $\phi_X(t) = e^{-|t|}$, t is real.

18. (a) State and prove Khintchine's WLLN under the first moment assumption.

    (b) What are the conditions under which a sequence of independent r.vs is said to follow the WLLN?

19. (a) State and prove the Lindberg-Levy CLT.

    (b) Show that it is a special case of the Lindberge-Feller form of CLT.

20. (a) How is the concept of stability of random variables applicable in the law of large numbers? State and prove Kolmogrov's SLLN.

    (b) Let $\{X_n\}$ be a sequence of i.i.d r.vs with $P[X_n = \pm 1] = \frac{1-2^{-n}}{2}$ and $P[X_n = \pm 2^n] = 2^{-n-1}$. Show that the SLLN holds for the sequence.

(3 × 8 = 24 Marks)

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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2022

Statistics

ST 521 : PROBABILITY THEORY – II

(2021 Admission)

Time : 3 Hours Max. Marks : 60

PART – A

Answer **any four** questions. Each question carries **3** marks.

1. Show that $\mu'_{(r)} = P_X^r(s)$ at $s = 1$, where $\mu'_{(r)}$ is the *r*th descending moment of a r.v *X* and $P_X^r(s)$ is the *r*th derivative of the probability generating function of *X*.

2. State the Hölder's inequality, mentioning the assumptions to be satisfied. When will it reduce to the Cauchy-Schwartz inequality?

3. Prove or disprove : $X_n \xrightarrow{a.s} X \leftrightarrow X_n \xrightarrow{P} X$.

4. Prove that if $E\left|X_n\right|^r \to 0$, then $X_n \xrightarrow{P} 0$.

5. Write the uniqueness theorem of characteristic function of a r.v.

6. State the Bochner's Theorem.

7. Check whether WLLN holds for the sequence $\{X_n, n \geq 1\}$ of random variables defined as $P(X_n = \sqrt{n}) = \frac{1}{2} = P(X_n = -\sqrt{n})$.

8. State the Linberg-Feller form of central limit theorem.

(4 × 3 = 12 Marks)

P.T.O.

PART – B

Answer **any three** questions. Each question carries **8** marks.

9. Show that Expectation is scale and location invariant.

10. Suppose X is an integer valued r.v for which the moments of all order exist. Then prove $P_X(s) = \sum_{k=0}^{o} E(X^{(k)}) \frac{(s-1)^k}{k!}$; $|s| \leq 1$ where $E(X^{(k)}) = E(X(X-1)$ ...$(X-k+1))$ is the kth descending factorial moment.

11. State and prove the Basic inequality.

12. (a) If $\{X_n, n \geq 1\}$ is a sequence of random variables where $X_n \xrightarrow{P} X$, then show that there exists a subsequence of it which converges almost sure to X.

    (b) State the Helly-Bray Theorem.

13. Prove or disprove: $X_n \xrightarrow{P} X \to X_n \xrightarrow{L} X$. Is the converse true? Justify.

14. Show that $X_n \xrightarrow{r} X \Rightarrow E|X_n|^r \to E|X|^r$.

(3 × 8 = 24 Marks)

PART – C

Answer **any three** questions. Each question carries **8** marks.

15. Show that characteristic function of a random variable is non-negative definite.

16. Prove that the characteristic function is real and even if and only if the corresponding probability distribution is symmetric about the origin.

17. State and prove the Fourier inversion theorem of characteristic function.

18. (a) State and prove the Chebychef's WLLN.

(b) State Kolmogrov's Three series theorem.

19. State and prove the Lindberg-Levi form of CLT.

20. (a) Show that every sequence of independent random variables with uniformly bounded variance obeys the SLLN.

(b) Check whether the following sequence of i.i.d r.vs $\{X_n\}$ defined as $P(X_n = \pm 1) = \frac{1-2^{-n}}{2}$ $P(X_n = \pm 2^n) = 2^{-n-1}$, obey the WLLN and the SLLN.

(3 × 8 = 24 Marks)
